§ Tangent planes & the differential of a map (do Carmo §2.4)

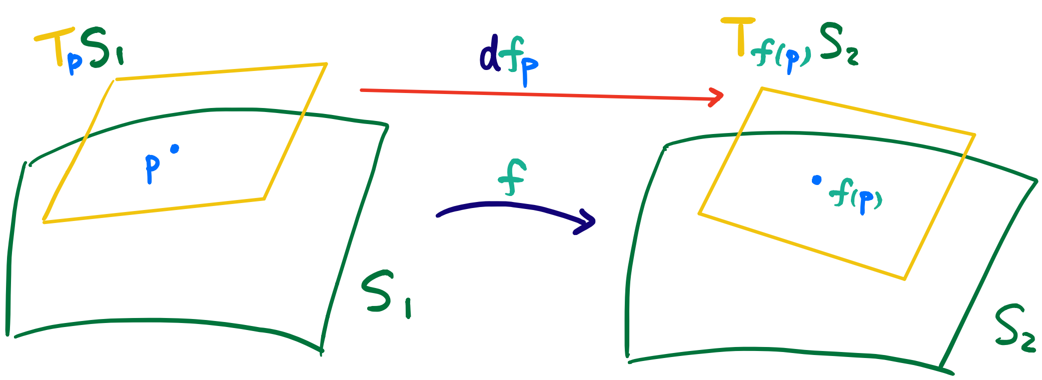
<u>Recall</u>: For a smooth map $f : \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $p \in \mathcal{U}$,

differential $\quad df_p : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad$ <u>linear map</u>

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ T_p\mathcal{U} & & T_{f(p)}\mathbb{R}^m \end{array}$$

If $f : S_1 \longrightarrow S_2$ is a smooth map between surfaces,

differential of f at $p \in S_1$ $\quad \boxed{df_p : T_pS_1 \longrightarrow T_{f(p)}S_2} \quad$ linear



<u>Question</u>: How to define it ?

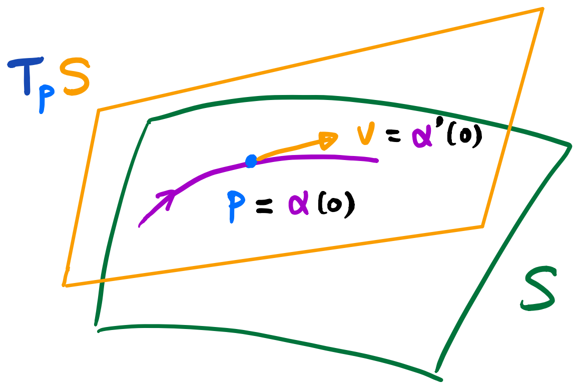
There are many (but equivalent) ways to define it.

We will use curves on surfaces.

Def$^{\underline{n}}$: The tangent plane of S at $p \in S$

not unique!

$$T_pS := \left\{ v \in \mathbb{R}^3 : \begin{array}{l} \exists \text{ smooth curve } \alpha : (-\varepsilon, \varepsilon) \to S \\ \text{s.t. } \alpha(0) = p \text{ and } \alpha'(0) = v \end{array} \right\}$$



Note:

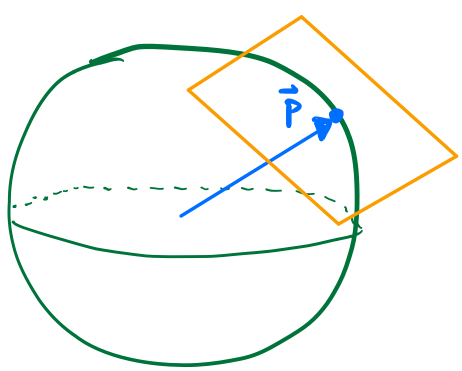
α may not be regular.

Example: $S = F^{-1}(a)$ for some regular value $a \in \mathbb{R}$ of a smooth $F : \underset{\text{open}}{O} \subseteq \mathbb{R}^3 \to \mathbb{R}$.

$p \in S$

Then, $$T_pS = (\underbrace{\nabla F|_p}_{\neq \vec{0}})^{\perp}$$ by "advanced calculus"

E.g. Sphere $S = \mathbb{S}^2(r) := \{ x^2 + y^2 + z^2 = r^2 \}$.



$T_pS \perp \vec{p}$ ← position vector at p

Exercise: Prove this!

We now define the differential of a real-valued function on surfaces.

Def^{n}: Let $f : S \to \mathbb{R}$ be a smooth function on a surface S, and $p \in S$. The **differential of f at p** is a map

$$df_p : T_pS \longrightarrow \mathbb{R}$$

defined as follows: for each $v \in T_pS$, choose any curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow S$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$

$$df_p(v) := \left.\frac{d}{dt}\right|_{t=0} f(\alpha(t))$$

Remark: We need to check that df_p is well-defined (i.e. independent of the choice of α) and that it is a linear map.

Similarly, we can define the differential of a smooth map between surfaces.

<u>Def$^{\underline{n}}$</u>: Let $f: S_1 \to S_2$ be a smooth map between surfaces.

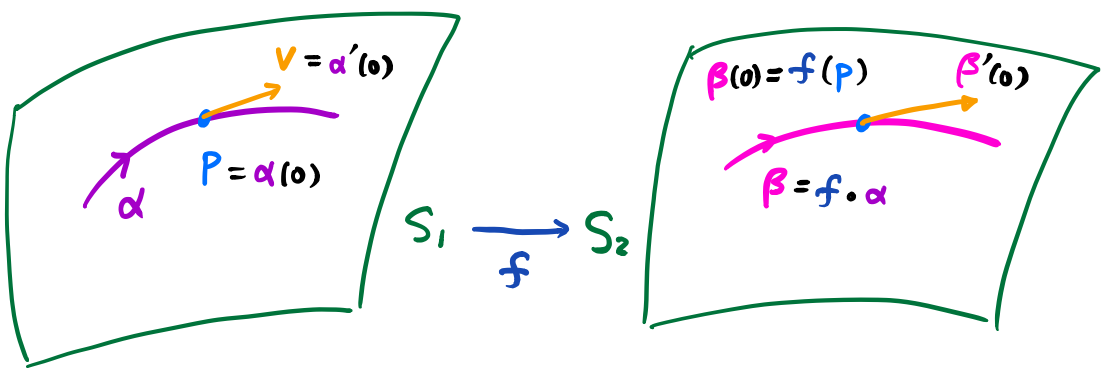
The differential of f at $p \in S_1$ is a map

$$df_p : T_p S_1 \longrightarrow T_{f(p)} S_2$$

defined as follows: for each $v \in T_p S_1$, choose any curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow S_1$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$

let $\beta = f \circ \alpha : (-\varepsilon, \varepsilon) \longrightarrow S_2$ with $\beta(0) = f(p)$

$$df_p(v) := \beta'(0) \in T_{f(p)} S_2$$



<u>Remark:</u> We need to check that df_p is well-defined (i.e. independent of the choice of α) and that it is a linear map between the tangent planes.

Recall that T_pS and $df_p : T_pS_1 \longrightarrow T_{f(p)}S_2$ are defined using curves on surfaces:

$$T_pS := \left\{ v \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} \exists\ \alpha : (-\varepsilon, \varepsilon) \to S \text{ s.t.} \\ \alpha(0) = p\ ,\ \alpha'(0) = v \end{array} \right\}$$

and $$df_p(\alpha'(0)) := (f \circ \alpha)'(0)$$ well-defined? linear?

<u>Lemma:</u> If $X : U \subseteq \mathbb{R}^2 \longrightarrow S$ is <u>ANY</u> parametrization s.t. $X(q) = p \in S$, then

$$\boxed{T_pS = dX_q(\mathbb{R}^2) = \text{span}\left\{ \frac{\partial X}{\partial u}, \frac{\partial X}{\partial v} \right\}}$$

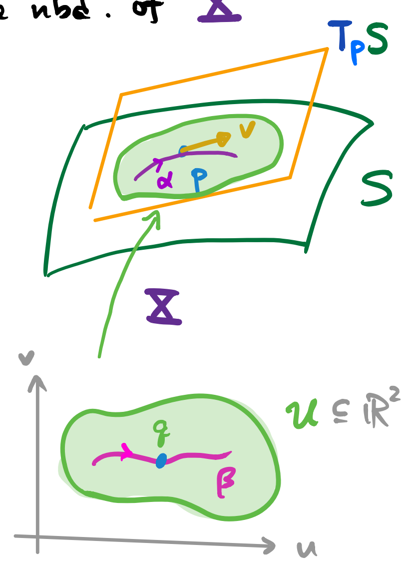
<u>Proof:</u> Take any $v \in T_pS$, by definition

$\exists\ \alpha : (-\varepsilon, \varepsilon) \longrightarrow S$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$

Restricting $\varepsilon > 0$ small if necessary, we can assume that α lies completely inside the coordinate nbd. of X

$\Rightarrow \exists\ \beta : (-\varepsilon, \varepsilon) \longrightarrow U \subseteq \mathbb{R}^2$ s.t.

$$\alpha = X \circ \beta$$



By Chain Rule,

$$v = \alpha'(0) = \underbrace{dX_q(\beta'(0))}_{\in\, dX_q(\mathbb{R}^2)}$$

Even more explicitly, we write

$$\beta(t) = (u(t), v(t)) \quad , \quad t \in (-\varepsilon, \varepsilon)$$

Then, by Chain Rule,

$$v = \alpha'(0) = \frac{d}{dt}\Big|_{t=0} X(u(t), v(t))$$

$$= \underbrace{u'(0)\, X_u\big|_q + v'(0)\, X_v\big|_q}_{\in \operatorname{span}\{X_u, X_v\}}$$

$\square$

Remark: 1) Since T_pS is defined independent of the choice of parametrizations, if X and X' are two parametrizations of S covering p, we have

$$\boxed{T_pS = \operatorname{span}\{X_u, X_v\} = \operatorname{span}\{X'_{u'}, X'_{v'}\}}$$

2) We will often write a tangent vector as

$$\boxed{v = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}}$$

as directional derivative $\quad v(f) := a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}$ where $f \in C^\infty(S)$

<u>Note</u>: Let $f : S \to \mathbb{R}$ be a smooth function. To see that $df_p : T_pS \to \mathbb{R}$ is well-defined & linear.

Let $v \in T_pS$, then $\exists$ unique $a, b \in \mathbb{R}$ s.t.

$$v = a\,\mathbf{X}_u + b\,\mathbf{X}_v$$

for some fixed parametrization $\mathbf{X}(u,v)$.

Write $\tilde{f}(u,v) = f \circ \mathbf{X}(u,v)$. Then, using notation as in previous lemma

$$df_p(v) := \left.\frac{d}{dt}\right|_{t=0} f(\alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} \tilde{f}(\beta(t))$$

$$= a \left.\frac{\partial \tilde{f}}{\partial u}\right|_q + b \left.\frac{\partial \tilde{f}}{\partial v}\right|_q$$

Thus, df_p is well-defined & linear.

Now, if $f : S_1 \to S_2$ is a smooth map between surfaces.

Take any parametrization of S_1 near p:

$$\begin{array}{ccccc} \mathbf{X} : & \mathcal{U} \subseteq \mathbb{R}^2 & \longrightarrow & S_1 \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & q & & p = \mathbf{X}(q) \end{array}$$

By previous lemma,

$$d\mathbf{X}_q : \mathbb{R}^2 \xrightarrow[\text{linear isomorphism}]{\cong} T_pS_1 \subseteq \mathbb{R}^3$$

$\rightsquigarrow$ $(d\mathbf{X}_q)^{-1} : T_pS_1 \longrightarrow \mathbb{R}^2$ exists.

Then one can check easily:

$$df_p = \underbrace{d(f \circ \mathbb{X})_q}_{\mathbb{R}^2 \to T_{f(p)}S_2 \subseteq \mathbb{R}^3} \circ \underbrace{(d\mathbb{X}_q)^{-1}}_{T_pS_1 \to \mathbb{R}^2}$$

well-defined & linear

$\square$

Properties of differential:

(1) Chain Rule : $\boxed{d(g \circ f)_p = dg_{f(p)} \circ df_p}$

(2) Inverse Function Theorem

Any smooth $f : S_1 \to S_2$ with

$$df_p : T_pS_1 \xrightarrow{\cong} T_{f(p)}S_2 \quad \text{linear isomorphism}$$

is a local diffeomorphism near p.

(3) If $f : S \to \mathbb{R}^m$ has $df_p = 0$ for all $p \in S$ and S is connected, then $f \equiv$ constant.

(3) If $f : S \to \mathbb{R}$ has a local max/min at $p \in S$ then $df_p = 0$.

Proof: Exercises!

In other words, all the familiar facts in calculus hold for functions defined on surfaces.